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Superfield formalism for the one loop effective action and CP(N) model in three dimensions

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Abstract: To obtain the one loop effective action for a given superfield theory, one encounters the notion such as the 'supertrace' of a differential operator on superspace. We develop, in a systematic way for the superspace of arbitrary dimension, a method to determine the supertrace precisely. We present a formula to express the supertrace explicitly as the superspace integral, which enables us to write the one loop effective action within the superfield formalism and still maintain the manifest supersymmetry. In the second part of the paper, we apply the result to a three dimensional $\mathcal{N} = 1$ supersymmetric CP(N) model in the auxiliary superfield formalism. The model contains a novel topological interaction term. We show in the large-N limit the one loop effective action is given by the supersymmetric Maxwell-Chern-Simons theory.

Keywords: Supersymmetric Effective Theories, Superspaces.

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Contents

1. Introduction and summary

There are two approaches to the path integral quantization of supersymmetric field theories. One is to work in the component formalism. The functional integral is straightforward, but supersymmetry is not manifest. The other is to go to the superspace [1] which contains the anti-commuting Grassmann coordinates as well as the usual spacetime coordinates. The ordinary fields and their functional integrals are replaced by the superfields and super functional integrals [2]. In consequence, we encounter the notions such as superdeterminant and supertrace, and these quantities are to be directly evaluated on superspace. Superpropagators and super Feynman rules naturally follow on superspace and the supersymmetric effective action can be *canonically* computed while keeping the supersymmetry manifest [3, 4].

In the present paper, for the superspace of arbitrary dimension, we develop a novel technique to determine precisely the supertrace of a differential operator on superspace, which arises in the computation of the one loop effective action *within the path integral formalism*. We present a formula to express the supertrace explicitly as the superspace integral, which enables us to write the one loop effective action within the superfield formalism and still maintains the manifest supersymmetry. In the second part of the paper, we apply the result to a three dimensional $\mathcal{N} = 1$ supersymmetric $\mathrm{CP}(N)$ model in the auxiliary gauge superfield formalism [5, 6, 7]. The model also contains a topological interaction term. Its bosonic sector is the higher derivative $\mathrm{CP}(N)$ model with the Wess-Zumino-Witten term and also the topological current term squared [8]. Since the theory has the interesting properties in the large-N limit such as its renormalizability and the Maxwell-Chern-Simons terms being dynamically induced at a nontrivial UV fixed point, it is worthwhile to check how the properties persist in the supersymmetric case too.

The organization of the paper is as follows. Section 2 contains our main result for the one loop effective action on the superspace of arbitrary dimension. We consider an arbitrary differential operator on the superspace, Δ, which generally depends on the superspace coordinates as well as their derivatives. The one loop effective action then corresponds to the "supertrace of the logarithm of it". We conceive a certain dual orthonormal basis for the Grassmannian coordinates and express the differential operator as well as the superfields of the given theory in terms of these basis. The differential operator then corresponds to a supermatrix, $\tilde{\Delta}$, while the superfields are matched to the Z_2-graded vectors on which the supermatrix acts. Moreover, the computation of the one loop effective action within the path integral formalism yields the supertrace of the logarithm of the supermatrix corresponding to the differential operator, $\mathrm{Str}(\ln\widetilde{\Delta})$. Following the very definition of the supertrace for the supermatrix, we explicitly compute this quantity. Firstly, thanks to the orthonormal property of the dual basis, the supermatrix corresponding to the product of two differential operators is identical to the product of the two corresponding supermatrices, and hence we obtain the crucial identity, $\mathrm{Str}(\ln\widetilde{\Delta}) = \mathrm{Str}(\widetilde{\ln\Delta})$. Secondly, we demonstrate how to manipulate the supertrace of the supermatrix corresponding to an operator on superspace in terms of the superspace integral with the operator itself rather than the supermatrix. Combining the two results, we obtain the final formula which expresses $\mathrm{Str}(\ln\widetilde{\Delta})$ as the superspace integral of the multi-commutators between the logarithm of the operator, $\ln\Delta$, and the Grassmann coordinates. Our result can be rewritten in an alternative fashion to resemble some known conventional expression for the supertrace. We believe our result clarifies the precise meaning of it.

In section 3, we apply the above result to the three dimensional U(1) gauged $\mathcal{N}=1$ supersymmetric CP(N) model which contains the gauge superfield and also a topological interaction term. Sticking to the superspace formalism, we solve the mass gap equation and compute the one-loop effective action. We show that in the large-N limit, the theory is renormalizable and the one loop effective action is given by the supersymmetric Maxwell-Chern-Simons theory plus its higher spacetime derivative generalization. In particular, the first order in the derivative expansion corresponds to the supersymmetric generalization of the extended topological massive electrodynamics model studied by Deser and Jackiw [9].

The appendix demonstrates the relation of the gauged CP(N) model to the supersymmetric higher derivative CP(N) model with the Wess-Zumino-Witten term [8].

2. One loop effective action — general analysis

We consider a generic superspace of arbitrary dimension, $d+N_f$, where the even part is a spacetime of dimension, d, and the odd part consists of Grassmann variables, ϑ^α, $\alpha=1,2,\ldots,N_f$. Without loss of generality we take the real basis so that $\vartheta^\alpha=(\vartheta^\alpha)^\dagger$ and let N_f be a even number.

On the superspace the superfield is of the generic form,

$$\Phi(z)=\phi(x)+\vartheta^\alpha\psi_\alpha(x)+\cdots+\vartheta^1\vartheta^2\cdots\vartheta^{N_f}F(x)=\sum_{n=0}^{N_f}\frac{1}{n!}\vartheta^{\alpha_1}\vartheta^{\alpha_2}\cdots\vartheta^{\alpha_n}\phi_{\alpha_1\alpha_2\cdots\alpha_n}(x)\,. \quad (2.1)$$

Its complex conjugate superfield reads

$$\Phi^{\dagger} = \sum_{n=0}^{N_f} \frac{1}{n!} \phi^{\dagger}_{\alpha_1\alpha_2\cdots\alpha_n} \vartheta^{\alpha_n} \cdots \vartheta^{\alpha_2}\vartheta^{\alpha_1} \,. \tag{2.2}$$

Any operator acting on the superfield is a function of the superspace coordinates and their derivatives, $\Delta(x^{\mu}, \vartheta^{\alpha}, \partial_{\nu}, \partial_{\beta})$. The typical manipulation to get the one loop effective action is to integrate out the complex scalar superfields,

$$\int D\Phi D\Phi^{\dagger}\, e^{\int dz^{d+N_f}\, \Phi^{\dagger}\Delta\Phi} = \frac{1}{\text{“ sdet }\Delta\text{”}} = e^{-\text{“ Str}(\ln\Delta)\text{”}} \,. \tag{2.3}$$

In the remaining of this section we show that the precise meaning of “ $\mathrm{Str}(\ln\Delta)$” is in fact,

$$\begin{aligned} \text{“ Str}(\ln\Delta)\text{”} &= \int dz^{d+N_f}\, \frac{1}{N_f!} \epsilon^{\alpha_1\alpha_2\cdots\alpha_{N_f}} \langle x|\{[\cdots\{[\ln\Delta, \vartheta^{\alpha_1}], \vartheta^{\alpha_2}\}\cdots]\vartheta^{\alpha_{N_f}}\}|x\rangle \\ &= \int dz^{d+N_f}\, \lim_{z'\rightarrow z} \langle z'|\ln\Delta|z\rangle = \int dz^{d+N_f}\, \lim_{\vartheta'\rightarrow\vartheta} \langle x|\ln\Delta\,\delta(\vartheta-\vartheta')|x\rangle \,, \end{aligned} \tag{2.4}$$

where $\delta(\vartheta-\vartheta') = (\vartheta^1-\vartheta'^1)(\vartheta^2-\vartheta'^2)\cdots(\vartheta^{N_f}-\vartheta'^{N_f})$. The second line resembles some known result in the literature [11], where the supertrace is simply given by $\int dz^{d+N_f}\,\langle z|\mathcal{O}|z\rangle$. Hence, our result clarifies the precise meaning of the conventional expression.
Proof. We first let all the possible independent products of the Grassmann coordinates,

$$\{\vartheta^{\alpha_1}\vartheta^{\alpha_2}\cdots\vartheta^{\alpha_n}\,,\ n=0,1,\ldots,N_f\} \equiv \{|M\rangle\}$$

be a basis for the ϑ^{α} expansion of the superfield with the dimension, 2^{N_f}, i.e. $M = 1,2,\ldots,2^{N_f}$. Then formally we can put

$$\Phi(z) = \sum_M |M\rangle\, \widetilde{\Phi}^M(x) \,, \tag{2.5}$$

and write an associated column vector,

$$\widetilde{\Phi} = (\widetilde{\Phi}^1, \widetilde{\Phi}^2, \ldots, \widetilde{\Phi}^M, \ldots)^T \,. \tag{2.6}$$

Now we define the dual orthonormal basis, $\{\langle K|\}$ by $\langle K|M\rangle = \delta_{KM}$. Explicitly it is of the form

$$\langle K| = \partial_1\partial_2\cdots\partial_{N_f}\vartheta^{N_f}\cdots\vartheta^2\vartheta^1\partial_{\beta_m}\cdots\partial_{\beta_2}\partial_{\beta_1} \,, \tag{2.7}$$

and the orthonormality reads

$$(\partial_1\partial_2\cdots\partial_{N_f}\vartheta^{N_f}\cdots\vartheta^2\vartheta^1\partial_{\beta_m}\cdots\partial_{\beta_2}\partial_{\beta_1})(\vartheta^{\alpha_1}\vartheta^{\alpha_2}\cdots\vartheta^{\alpha_n}) = n!\,\delta_m^{\,n}\,\delta_{\beta_1}^{[\alpha_1}\delta_{\beta_2}^{\alpha_2}\cdots\delta_{\beta_n}^{\alpha_n]} \,, \tag{2.8}$$

where on the right hand side $[\alpha_1\alpha_2\cdots\alpha_n]$ denotes the total anti-symmetrization having the “strength one” or the multiplication factor, $1/n!$. The usual completeness relation follows straightforwardly,

$$\sum_K |K\rangle\langle K| = \text{Identity} \,. \tag{2.9}$$

Using the orthonormal basis above we can express any operator acting on the superfields,

$$\mathcal{O}(x^\mu, \vartheta^\alpha, \partial_\nu, \partial_\beta) = |M\rangle \widetilde{\mathcal{O}}_{MK}(x^\mu, \partial_\nu)\langle K| \,. \tag{2.10}$$

The crucial merit of conceiving the above orthonormal dual basis is as follows. The product of operators can be also rewritten in terms of the basis. From the orthonormality we get

$$\mathcal{O}_1\mathcal{O}_2 = |M\rangle \widetilde{\mathcal{O}_1\mathcal{O}_2}_{MK}\langle K| \,, \qquad \widetilde{\mathcal{O}_1\mathcal{O}_2}_{MK} = \langle M|\mathcal{O}_1\mathcal{O}_2|K\rangle = \widetilde{\mathcal{O}}_{1\,ML}\widetilde{\mathcal{O}}_{2\,LK} \,. \tag{2.11}$$

Namely, the supermatrix, $\widetilde{\mathcal{O}}_{MK} = \langle M|\mathcal{O}|K\rangle$, gives a good representation for the product of the operators.

The supertrace of the supermatrix, $\widetilde{\mathcal{O}}$, can be now expressed as a superspace integration of the operator itself,

$$\begin{aligned}
\mathrm{Str}\,\widetilde{\mathcal{O}} &= \sum_M (-1)^{\#(M)}\langle M|\mathcal{O}|M\rangle \\
&= \mathrm{tr}\left[\sum_{n=0}^{N_f} \frac{(-1)^n}{n!}\partial_1\partial_2\cdots\partial_{N_f}\vartheta^{N_f}\cdots\vartheta^2\vartheta^1\partial_{\alpha_n}\cdots\partial_{\alpha_2}\partial_{\alpha_1}\mathcal{O}\vartheta^{\alpha_1}\vartheta^{\alpha_2}\cdots\vartheta^{\alpha_n}\right] \\
&= \int d\vartheta^{N_f}\,\mathrm{tr}\left[\sum_{n+m=N_f}\frac{(-1)^n}{n!\,m!}\epsilon^{\alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_m}\vartheta^{\alpha_1}\vartheta^{\alpha_2}\cdots\vartheta^{\alpha_n}\mathcal{O}\vartheta^{\beta_1}\vartheta^{\beta_2}\cdots\vartheta^{\beta_m}\right] \\
&= \int d\vartheta^{N_f}\;\mathrm{tr}\left[\frac{1}{N_f!}\epsilon^{\alpha_1\alpha_2\alpha_3\cdots\alpha_{N_f}}\{[\cdots[\{[\mathcal{O},\vartheta^{\alpha_1}],\vartheta^{\alpha_2}\},\vartheta^{\alpha_3}]\cdots]\vartheta^{\alpha_{N_f}}\}\right].
\end{aligned} \tag{2.12}$$

Here 'tr' denotes the trace over the spacetime coordinates, $\epsilon^{\alpha_1\alpha_2\cdots\alpha_{N_f}}$ is the totally anti-symmetric tensor such that $\epsilon^{12\cdots N_f} = 1$, and our convention for the integration over the Grassmann coordinates is $\int d\vartheta^{N_f}\vartheta^1\vartheta^2\cdots\vartheta^{N_f} = 1$.

We also have

$$\mathcal{O}\Phi = \sum_{M,K}|M\rangle\widetilde{\mathcal{O}}_{MK}\widetilde{\Phi}^K \,, \tag{2.13}$$

and

$$\int d\vartheta^{N_f}\Phi^\dagger\Psi = \sum_{n+m=N_f}\frac{1}{n!m!}\,\phi^\dagger_{\alpha_1\alpha_2\cdots\alpha_n}\epsilon^{\alpha_n\cdots\alpha_2\alpha_1\beta_1\beta_2\cdots\beta_m}\psi_{\beta_1\beta_2\cdots\beta_m} \equiv \widetilde{\Phi}^\dagger\widetilde{\mathcal{E}}\widetilde{\Psi} \,. \tag{2.14}$$

Now the one loop effective action (2.3) reads, from (2.11) and $\mathrm{sdet}\,\widetilde{\mathcal{E}} = 1$,

$$\begin{aligned}
\ln\left(\int D\Phi D\Phi^\dagger e^{\int dz^{d+N_f}\,\Phi^\dagger\Delta\Phi}\right) &= -\ln\left[\mathrm{sdet}(\widetilde{\mathcal{E}}\widetilde{\Delta})\right] = -\,\mathrm{Str}(\ln\widetilde{\Delta}) = -\,\mathrm{Str}(\widetilde{\ln\Delta}) \\
&= -\int dz^{d+N_f}\,\frac{1}{N_f!}\epsilon^{\alpha_1\alpha_2\cdots\alpha_{N_f}}\,\times \\
&\qquad \times\,\langle x|\{[\cdots\{[\ln\Delta,\vartheta^{\alpha_1}],\vartheta^{\alpha_2}\}\cdots]\vartheta^{\alpha_{N_f}}\}|x\rangle \,.
\end{aligned} \tag{2.15}$$

Instead of using the dual orthonormal basis, if we consider the expansion of an operator by the ordinary derivatives for the Grassmann coordinates,

$$\mathcal{O}(z,\partial_\mu,\partial_\alpha)=\sum_{n=0}^{N_f}\frac{1}{n!}\,\mathcal{O}(z,\partial_\mu)^{\alpha_1\alpha_2\cdots\alpha_n}\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_n}\,, \tag{2.16}$$

it is straightforward to check that *only the highest order component contributes to the supertrace*. Thus, formally introducing $|\vartheta\rangle$ such that

$$\langle\vartheta'|\vartheta\rangle=\delta(\vartheta-\vartheta')=(\vartheta^1-\vartheta'^1)(\vartheta^2-\vartheta'^2)\cdots(\vartheta^{N_f}-\vartheta'^{N_f})\,, \tag{2.17}$$

we get

$$\mathrm{Str}\,\widetilde{\mathcal{O}}=\int dz^{d+N_f}\,\langle x|\mathcal{O}(z,\partial_\mu)^{N_f\cdots 21}|x\rangle=\int dz^{d+N_f}\,\lim_{z'\to z}\langle z'|\mathcal{O}|z\rangle\,. \tag{2.18}$$

This completes our proof. □

Recently one of the authors showed that every superfield theory can be described by a dual supermatrix model [10]. It will be interesting to see how our result persists in the supermatrix side.

3. $\mathcal{N}=1$ supersymmetric $\mathrm{CP}(N)$ model in three dimensions

In this section, we consider a three dimensional supersymmetric $\mathcal{N}=1$ $\mathrm{CP}(N)$ model with a topological interaction term. Our notation is as follows. The spacetime metric is $\eta=\mathrm{diag}(+--)$, and the gamma matrices are all imaginary given by the Pauli matrices, $\gamma^0=\sigma^2$, $\gamma^1=i\sigma^3$, $\gamma^3=i\sigma^1$ [11]. The spinors are real $\theta_\alpha={\theta_\alpha}^\dagger$, $\alpha=1,2$, and the adjoint of the spinor is pure imaginary, $\bar\theta=\theta^\dagger\gamma^0$. The charge conjugation matrix, $C_{\alpha\beta}$, satisfies

$$C^{-1}\gamma^\mu C=-\gamma^{\mu T}\,,\qquad C=-C^T=C^{-1}\,,\qquad C=\gamma^0\,. \tag{3.1}$$

Using the covariant derivatives given by

$$D_\alpha=\frac{\partial}{\partial\bar\theta^\alpha}-i(\gamma^\mu\theta)_\alpha\frac{\partial}{\partial x^\mu}\,,\qquad \bar D^\alpha=D_\beta C^{-1\beta\alpha}=-\frac{\partial}{\partial\theta_\alpha}+i(\bar\theta\gamma^\mu)^\alpha\frac{\partial}{\partial x^\mu}\,, \tag{3.2}$$

we consider the following action of the $\mathcal{N}=1$ supersymmetric $\mathrm{CP}(N)$ model [5, 6, 7] in three dimensions with a topological interaction;

$$\mathcal{S}=\int dx^3d\theta^2\left[\frac{N}{2g}\Big(\bar\nabla^\alpha\bar\Phi_i\nabla_\alpha\Phi_i+2\Sigma(\bar\Phi_i\Phi_i-1)\Big)-i\frac{\kappa}{4}N\Big(\bar F^\alpha\bar\Phi_iD_\alpha\Phi_i-\bar D^\alpha\bar\Phi_i\Phi_iF_\alpha\Big)\right]. \tag{3.3}$$

The first two terms corresponds to the supersymmetric $\mathrm{CP}(N)$ model and the last two terms describes the topological interaction [8]. In the above action, Φ_i, $\bar\Phi_i={\Phi_i}^\dagger$, $i=1,\dots,N$ are complex superfields,

$$\Phi_i(x,\theta)=\phi_i(x)+\bar\theta\psi_i(x)+\frac{1}{2}\bar\theta\theta F_i(x)\,, \tag{3.4}$$

and Σ is a real superfield serving as a Lagrange multiplier,

$$\Sigma(x,\theta)=\sigma(x)+\bar\theta\xi(x)+\frac{1}{2}\bar\theta\theta\alpha(x)\,. \tag{3.5}$$

The gauge covariant derivatives are given by

$$\nabla_\alpha = D_\alpha + iA_\alpha \,, \qquad \bar{\nabla}^\alpha = \bar{D}^\alpha - i\bar{A}^\alpha \,, \tag{3.6}$$

with a real spinor superfield, A_α,

$$A_\alpha = \chi_\alpha + i(\gamma^\mu\theta)_\alpha A_\mu + \theta_\alpha D + \frac{1}{2}\bar{\theta}\theta\omega_\alpha \,. \tag{3.7}$$

A_μ is the usual U(1) auxiliary gauge field and $\bar{\nabla}^\alpha\bar{\Phi} = (\nabla_\beta\Phi)^\dagger C^{-1\beta\alpha}$.

The above action is invariant under the U(1) gauge transformation given by

$$\Phi \to e^{i\Lambda}\Phi \,, \qquad A_\alpha \to A_\alpha + D_\alpha\Lambda \,, \tag{3.8}$$

where Λ is a real scalar superfield,

$$\Lambda = b + \bar{\theta}f + \frac{1}{2}\bar{\theta}\theta a \,. \tag{3.9}$$

In the Wess-Zumino gauge, $f_\alpha = -\chi_\alpha$, $a = -D$,

$$A_\alpha(x,\theta) = i(\gamma^\mu\theta)_\alpha A_\mu + \frac{1}{2}\bar{\theta}\theta\omega_\alpha \,, \tag{3.10}$$

and the real spinor superfield strength, F_α, takes the form,

$$F_\alpha = -\bar{D}^\beta D_\alpha A_\beta = \omega_\alpha + (F_{\mu\nu}\gamma^{\mu\nu}\theta)_\alpha - i\frac{1}{2}\bar{\theta}\theta(\gamma^\mu\partial_\mu\omega)_\alpha \,, \tag{3.11}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that $\bar{F}^\alpha = F_\beta C^{-1\beta\alpha}$ and F_α is gauge invariant due to the identity, $\bar{D}^\beta D_\alpha D_\beta = 0$. The action is also invariant under the global SU(N) symmetry, $\Phi_i \to U_i{}^j\Phi_j$.

Essentially the above action is the supersymmetric generalization of the bosonic model one of the authors studied previously [8]. It is demonstrated in the appendix that the spinor superfield, A_α, can be eliminated from the action, (3.3), using the equation of motion as in the case of the bosonic model, and the resulting action corresponds to the supersymmetric generalization of the higher derivative CP(N) model with the Wess-Zumino-Witten term.

3.1 The effective action and the gap equation

In order to obtain the effective action and gap equation, we integrate out the complex scalar superfield, Φ. We first rewrite the action (3.3) in the gaussian form as

$$\mathcal{S} = \int dz^5 \left[\frac{N}{2g}\bar{\Phi}_i\Delta\Phi_i - \frac{N}{g}\Sigma\right] , \tag{3.12}$$

with a differential operator, Δ,

$$\Delta = -\bar{D}D - i(\bar{D}A) - 2i\bar{A}D + \bar{A}A - i\kappa g\bar{F}D + 2\Sigma \,. \tag{3.13}$$

Introducing the external source J_i for $\bar{\Phi}_i$ (and similarly for $\bar{J}_i$ for Φ_i) given by

$$J_i = n_i + \bar{\theta}\eta_i + \frac{1}{2}\bar{\theta}\theta j_i \,, \tag{3.14}$$

we obtain the generating functional,

$$Z[J_i,\bar{J}_i]=\int\mathcal{D}\Phi_i\mathcal{D}\bar{\Phi}_i\mathcal{D}A\mathcal{D}\bar{A}\mathcal{D}\Sigma\exp\left[i\int dz^5\left(\frac{N}{2g}\bar{\Phi}_i\Delta\Phi_i-\frac{N}{g}\Sigma+\frac{1}{2}\bar{J}_i\Phi_i+\frac{1}{2}\bar{\Phi}_iJ_i\right)\right]. \tag{3.15}$$

Integrating out the scalar superfields Φ_i, $\bar{\Phi}_i$ yields

$$Z[J_i,\bar{J}_i]=\int\mathcal{D}A\mathcal{D}\bar{A}\mathcal{D}\Sigma e^{iW_{\text{eff}}}\,,W_{\text{eff}}=iN\text{“}\,\mathrm{Str}(\ln\Delta)\text{”}-\int dz^5\,\frac{N}{g}\Sigma-\frac{g}{2N}\int dz^5\,\bar{J}_i\Delta^{-1}J_i\,, \tag{3.16}$$

where “ $\mathrm{Str}(\ln\Delta)$” can be explicitly read off from (2.4), as done in the next subsection 3.2.

Here we approximate (3.16) by the large-N saddle point method, and taking the Legendre transformation of W_{eff}, as usual, we get the effective action in the leading order,

$$\mathcal{S}_{\text{eff}}(\Phi_i,\bar{\Phi}_i,\Sigma)=iN\text{“}\,\mathrm{Str}(\ln\Delta)\text{”}+\int dz^5\frac{N}{2g}\left(\bar{\Phi}_i\Delta\Phi_i-2\Sigma\right). \tag{3.17}$$

Let us turn to the gap equation which can be derived from the stationary conditions of the effective potential. Taking advantage of the global U(N) symmetry, we write expectation values of Φ_i and $\bar{\Phi}_i$ as

$$(\langle\Phi_1\rangle,\langle\Phi_2\rangle,\dots,\langle\Phi_N\rangle)=(0,0,\dots,\sqrt{g}\,V)\,, \tag{3.18}$$

and $\langle\Sigma\rangle$ for that of Σ. Turning off the spinor superfield, $A_\alpha=0$, we obtain the effective potential

$$\begin{aligned}U_{\text{eff}}&\equiv-\Omega^{-1}\mathcal{S}_{\text{eff}}\\&=-N\int d\theta^2\left(\bar{V}V-\frac{1}{g}\right)\langle\Sigma\rangle-i\frac{1}{2}N\int d\theta^2\langle x|\left\{\bar{\theta}^\alpha,[\theta_\alpha,\ln(-\bar{D}D+2\langle\Sigma\rangle)]\right\}|x\rangle\\&=-N\int d\theta^2\left(\bar{V}V-\frac{1}{g}\right)\langle\Sigma\rangle-iN\int d\theta^2\lim_{z'\to z}\langle z'|\ln(-\bar{D}D+2\langle\Sigma\rangle)|z\rangle\,,\end{aligned} \tag{3.19}$$

where Ω is the three-dimensional spacetime volume.

The stationary conditions for the effective potential read

$$\begin{aligned}\bar{V}\langle\Sigma\rangle=V\langle\Sigma\rangle&=0\,,\\ \bar{V}V-\frac{1}{g}+\frac{i}{(2\pi)^3}\int dk^3\,\frac{1}{k^2-\langle\Sigma\rangle^2+i\varepsilon}&=0\,.\end{aligned} \tag{3.20}$$

The UV divergence appears in the second formula and with the momentum cut-off Λ, it becomes

$$\bar{V}V-\frac{1}{g}+\frac{\Lambda}{2\pi^2}-\frac{\langle\Sigma\rangle}{4\pi}=0\,. \tag{3.21}$$

Now we introduce an arbitrary scale parameter, μ, and a renormalized coupling constant, g_r, which satisfy

$$\bar{V}V-\frac{1}{g_r}+\frac{\mu}{2\pi^2}-\frac{\langle\Sigma\rangle}{4\pi}=0\,. \tag{3.22}$$

Then, the renormalized coupling constant and the bare coupling constant are related by

$$\frac{1}{g}=\frac{1}{g_r}+\frac{\Lambda}{2\pi^2}-\frac{\mu}{2\pi^2}\,. \tag{3.23}$$

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With the introduction of the dimensionless coupling u defined by $u \equiv \Lambda g$, we obtain the β-function as

$$\beta(u) \equiv \Lambda \frac{du}{d\Lambda} = u\Big(1 - \frac{u}{2\pi^2}\Big) . \tag{3.24}$$

This β-function shows a nontrivial UV fixed point at $u = 2\pi^2 \equiv u^*$. In the continuum limit, $\Lambda \to \infty$, we have $u \to u^*$.

From (3.20) and (3.24), we find that there are two phases:

- (i) SU(N) symmetric phase for $u > 2\pi^2$,

$$V = \bar{V} = 0 , \tag{3.25}$$

$$\langle \Sigma \rangle = \langle \sigma \rangle + \bar{\theta}\langle \xi \rangle + \frac{1}{2}\bar{\theta}\theta\langle \alpha \rangle \equiv m . \tag{3.26}$$

- (ii) SU(N) $\to$ SU($N-1$) $\times$ U(1) broken phase for $u < 2\pi^2$,

$$\begin{aligned} V &= \frac{1}{\sqrt{g}}\left[\langle \phi_N \rangle + \bar{\theta}\langle \psi_N \rangle + \frac{1}{2}\bar{\theta}\theta\langle F_N \rangle\right] \equiv v \neq 0 , \\ \bar{V} &= \frac{1}{\sqrt{g}}\left[\langle \bar{\phi}_N \rangle + \langle \bar{\psi}_N \rangle\theta + \frac{1}{2}\bar{\theta}\theta\langle \bar{F}_N \rangle\right] \equiv \bar{v} \neq 0 , \\ \bar{V}V &= \Lambda\Big(\frac{1}{u} - \frac{1}{2\pi^2}\Big) , \\ \langle \Sigma \rangle &= 0 . \end{aligned} \tag{3.27}$$

Note that in both phases, from $\langle \xi \rangle = \langle \bar{\psi}_N \rangle = \langle \psi_N \rangle = 0$ and $\langle \alpha \rangle = \langle \bar{F}_N \rangle = \langle F_N \rangle = 0$, the supersymmetry is unbroken. For the symmetric phase, eq. (3.21) becomes

$$\left(\frac{1}{u} - \frac{1}{u^*}\right)\Lambda + \frac{m}{4\pi} = 0 . \tag{3.28}$$

This gap equation is identical to that in the bosonic theory [6, 8], and the mass scale, m, is an cutoff independent parameter of the theory.

3.2 One loop effective action and renormalization

In this subsection, we consider the symmetric phase and redefine Σ as

$$\Sigma = m + \Sigma' . \tag{3.29}$$

After some manipulation for the one loop effective action using the superfield formalism developed in section 2, we obtain the following large-N effective action up to the quadratic terms,

$$\begin{aligned} \mathcal{S}_{\text{eff}} = N \int dz^5 \Bigg[& \frac{1}{2g}\bar{\Phi}(-\bar{D}D + 2m - i(\bar{D}A) - 2i\bar{A}D + \bar{A}A - i\kappa g\bar{F}D + 2\Sigma')\Phi - \\ & - \left(\frac{1}{u} - \frac{1}{u^*}\right)\Lambda(m\delta^{(2)}(\theta) + \Sigma') - \frac{m}{4\pi}\Sigma' - \frac{m^2}{8\pi}\delta^{(2)}(\theta) - \\ & - \frac{1}{2}\Sigma'\Pi_{\Sigma'}(-i\partial)\Sigma' - \frac{1}{16}\bar{F}\Pi_1(-i\partial)F + \frac{m}{4}\bar{A}\Pi_1(-i\partial)F - \\ & - \frac{1}{2}\kappa g\bar{A}\Pi_2(-i\partial)F - \frac{1}{8}(\kappa g)^2\bar{F}\Pi_2(-i\partial)F \Bigg] , \end{aligned} \tag{3.30}$$

where

$$\Pi_{\Sigma'}(p) = \frac{\bar{D}(p,\theta)D(p,\theta)+4m}{8\pi\sqrt{-p^2}}\arctan\left(\frac{\sqrt{-p^2}}{2m}\right),$$
$$\Pi_1(p) = \frac{1}{4\pi\sqrt{-p^2}}\arctan\left(\frac{\sqrt{-p^2}}{2m}\right),$$
$$\Pi_2(p) = \frac{\Lambda}{2\pi^2}-\frac{m}{4\pi}+\frac{p^2-2m\not{p}}{8\pi\sqrt{-p^2}}\arctan\left(\frac{\sqrt{-p^2}}{2m}\right). \tag{3.31}$$

Note that in the above action, the terms containing $\delta^{(2)}(\theta)$ come from the effective potential (3.19), and the Λ dependent term in the second line becomes finite through the gap equation (3.28). Also, the derivative expansion in the third and fourth lines are manifest supersymmetric because they are all given in term of $(\bar{D}D)^2 = -4\partial_\mu\partial^\mu = 4p^2$.

As in the bosonic case [8], the above action is renormalizable in spite of the linear divergence term. Introducing the Z factor for the coupling constant

$$Z^{-1} \equiv \frac{g_r}{g} = 1+\frac{u_r}{u^*}\left(\frac{\Lambda}{\mu}-1\right), \qquad u_r \equiv \mu g_r\,, \tag{3.32}$$

and redefining $\Phi \equiv \sqrt{Z}\Phi_r$ in order to cancel the Z factor from the coupling constant renormalization, we recast the kinetic term as

$$\frac{1}{g}\bar{\Phi}(-\bar{D}D+2m)\Phi = \frac{1}{g_r}\bar{\Phi}_r(-\bar{D}D+2m)\Phi_r\,. \tag{3.33}$$

This makes sure the kinetic term is UV finite by itself. The spinor superfield, A_α, and the real superfield, Σ', do not need any wave function renormalization. The induced supersymmetric Maxwell-Chern-Simons term contains a linear divergence in $\Pi_2(p)$. However, the β-function (3.24) tells us that the dimensionless coupling $u \equiv g\Lambda$ goes to the UV fixed point, $u^* = 2\pi^2$, as the cutoff Λ goes to the infinity. Therefore, in the continuum limit, $\Lambda\to\infty$, we obtain the UV finite result, $g\Pi_2(p)\to 1$, $g^2\Pi_2(p)\to 0$, including the supersymmetric Maxwell-Chern-Simons terms,

$$-\frac{1}{16}\bar{F}\Pi_1(-i\partial)F+\frac{m}{4}\bar{A}\Pi_1(-i\partial)F-\frac{1}{2}\kappa\bar{A}F\,. \tag{3.34}$$

After all, the one loop effective action (3.30) leads, in the continuum limit, to a scalar superfield theory coupled to the derivatively expanded Maxwell-Chern-Simons theory;

$$\begin{aligned}\mathcal{S}_{\text{eff}} = N\int dz^5\Big[&\frac{1}{2g_r}\bar{\Phi}_r\left(-\bar{D}D+2m-i(\bar{D}A)-2i\bar{A}D+\bar{A}A-i\kappa g_r\bar{F}D+2\Sigma'\right)\Phi_r+\\ &+\frac{m^2}{8\pi}\delta^{(2)}(\theta)-\frac{1}{2}\Sigma'\Pi_{\Sigma'}(-i\partial)\Sigma'-\frac{1}{16}\bar{F}\Pi_1(-i\partial)F+\\ &+\frac{m}{4}\bar{A}\left(\Pi_1(-i\partial)-\frac{2\kappa}{m}\right)F\Big].\end{aligned} \tag{3.35}$$

At the lowest order of the expansion in ∂/m, the supersymmetric Maxwell-Chern-Simons theory is induced with the coefficient of the Chern-Simons term given by $\frac{1}{32\pi}-\frac{1}{2}\kappa$.

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The origin of the parity-violating Chern-Simons term with the coefficient, $1/32\pi$, is due to the parity violating fermion mass term in the first line of the action (3.30) [13], while the topological interaction term shifts the coefficient by an amount of $-1/2\kappa$. It is interesting to note that the next order of the expansion of the Chern-Simons term yields the supersymmetric generalization of the higher derivative Chern-Simons extensions, $\sim m^{-2}\bar{A}(\bar{D}D)^2F$ [9], and with the choice of the coefficient $\kappa \equiv 1/16\pi$, the gauge sector of the effective action becomes the supersymmetric generalization of the extended topological massive electrodynamics model of ref. [9]

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A. Supersymmetric generalization of the higher derivative CP(N) model

Here we show that the spinor superfield, A_α, can be eliminated from the action, (3.3), using the equation of motion, and the resulting action corresponds to the supersymmetric generalization of the higher derivative CP(N) model [8].

The equation of motion for the spinor superfield reads from

$$\delta_A \mathcal{S} = \int d^3x d^2\theta \frac{N}{2g}\left[\delta\bar{A}^\alpha\left(A_\alpha - i\Phi^\dagger D_\alpha\Phi - \frac{g\kappa}{2}\bar{D}^\beta D_\alpha(i\Phi^\dagger D_\beta\Phi)\right) + \text{c.c.}\right], \tag{A.1}$$

so that

$$A_\alpha = J_\alpha + \frac{g\kappa}{2}t_\alpha\,, \tag{A.2}$$

where $J_\alpha = i\Phi^\dagger D_\alpha\Phi$, $t_\alpha = \bar{D}^\beta D_\alpha J_\beta$ are the current density and the topological current density respectively, satisfying $\bar{D}^\alpha t_\alpha = 0$.

Substituting the expression into the action we obtain

$$\mathcal{S}' = \frac{N}{2g}\int d^3x d^2\theta\left(\bar{D}^\alpha\Phi^\dagger D_\alpha\Phi - \bar{J}^\alpha J_\alpha - \frac{g^2\kappa^2}{4}\bar{t}^\alpha t_\alpha\right) + \mathcal{S}_{WZW}\,. \tag{A.3}$$

The third term inside the bracket is the supersymmetric generalization of the fourth order derivative term [8], while $\mathcal{S}_{WZW}$ is the supersymmetric Wess-Zumino-Witten term in three dimensions [12],

$$\mathcal{S}_{WZW} = -\frac{\kappa}{4}N\int d^3x d^2\theta\left(\bar{t}^\alpha J_\alpha + \bar{J}^\alpha t_\alpha\right). \tag{A.4}$$

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